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# Application of Game Theory to Pricing of Participating Deferred Annuity

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**Abstract:** We study pricing models for a participating deferred annuity. Game theory is used to formulate different pricing models based on customers' preference concerning benefits and risks. The objective is to maximize social welfare. Value at Risk (VaR) under multi-stage stochastic processes is applied to measure credit risk and its calculation is discussed. Monte Carlo simulation and stochastic optimization are used to find optimal solutions for price and dividend rate. [Key words: participating annuity, game theory, stochastic process]

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## INTRODUCTION

Game theory is an area of applied mathematics that studies strategic interactions among game participants (players), where players choose different actions in an attempt to maximize their returns. Although it may appear similar to decision theory, game theory studies decisions that are made in an environment where various players interact. In other words, game theory studies choice of optimal behavior when the costs and benefits of each option are not fixed, but depend upon the choices of other individuals. Pricing is a game, as defined by game theory, because a firm's

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success depends not only on the price charged, but also on how customers and competitors respond to it. In the past decade we have seen an explosion of studies of customers and their preferences by the life insurance industry. Many companies now have a significant ability to gauge customers' willingness to pay for their products. The question discussed in this article deals with using a cooperative game. In cooperative games, binding agreements are allowed, the players may form coalitions, and the outcome is chosen based on certain fairness criteria that both bargainers agree on (Cao, Shen, Milito, and Wirth, 2002) to establish a model for selecting the optimal participating strategy and pricing strategy, which realizes the objective of maximizing the sum of the values of both insurers and customers (a criterion that both parties agree on). We formulate several different price functions based on customers' different preferences with respect to benefits and risks, and establish different objective functions for each price function. The advantage of this framework is that we have a precise mathematical characterization of the solutions and their properties. Insurance pricing is a special pricing problem where uncertainty and risk are particularly important. We phrase it as a problem in game theory and research effective ways to find optimal solutions. We address a two-person decision problem under uncertainty, integrating game theory and stochastic optimization techniques and considering the effects of different price functions on decision-making.

Since Value at Risk (VaR), defined as the loss that can occur over a given period, at a given confidence level, is linked directly to credit risk and can be directly observed by insurers and customers, it has been widely adopted as a measurement of credit risk in recent years (notably in banking regulation). In this article, we use VaR as a measure of credit risk. Furthermore, we integrate stochastic rate of return and multi-stage accumulated investment in the established models. Since a deferred annuity is a long-term contract, and typically involves an investment lasting up to several decades, we believe that risk models must incorporate a stochastic structure of interest rate and cumulative investment.

## LITERATURE REVIEW

Participating life insurance is an important product with a sizable market share in the life insurance marketplace. The large part of the existing literature in this area has been focused on equity-linked contracts (see: Brennan and Schwartz, 1976, 1979; Bacinello and Orut, 1993; Milevsky and Posner, 2001; Bacinello and Persson, 2002). There also have been some

articles studying the fair value of participating life insurance and its risk management (e.g. Briys and de Varenne, 1994, 1997; Grosen and Jørgensen, 2000, 2002; Barbarin and Devolder, 2005). Most of them are based on Merton's (1974) option pricing approach and in the context of whole life insurance contracts. Although there are many papers dealing with game theory, the literature with applications of game theory in insurance is very limited. Borch (1962) has studied the problem of determining the "correct" premium rate for sub-groups of an insurance group using game theory. Schlesinger (1984) has studied the problem of a risk-neutral insurer and a risk-averse individual who bargain over the terms of an insurance contract.

## ASSUMPTIONS

Using the standard actuarial notation (Bowers et al., 1997), we denote by  $T(x)$  the random future lifetime of an insured aged  $x$ . As it is standard in the actuarial theory, we assume that the probability density function  $f_T$  of  $T(x)$  exists. Also,  ${}_t p_x$  is the probability that the insured is still alive at time  $t$ . Furthermore,  $T(x)$  is assumed to be independent of all financial random variables describing rates of return in this paper. We denote the maturity time of the deferred annuity contract under consideration by  $n$ . We assume that the premium is paid at the beginning of the year and the paying term is  $m$  (where  $m < n$ ). The financial market is assumed perfectly competitive, frictionless, and free of arbitrage opportunities. But the insurance market is incomplete. This means that the prices of insurance products are not unique but depend on the agents' attitudes towards risk and competitive power of insurance companies, corresponding elements of competition including expenses, interest rate, and rates of return. Moreover, all agents are assumed to be rational and non-satiated, and to share the same information. Let  $C_t$  denote the benefit payable to the insured if alive at time  $t$ , with  $t = m + 1, \dots, n$ , which is a quantity dependent on the value of the invested portfolio. We also assume that the volatility of interest rates is constant. The premium income at the beginning of each year is invested in a variable investment portfolio, a mutual fund. Let  $D_{jt}^1$  be the market value of the cumulative investment fund at time  $t$  with initial actuarial present value of  $D_{j0}^1 = P 1_{\{T(x) > j\}} e^{-r_j j} = {}_j p_x P e^{-r_j j}$   $j = 0, 1, 2, \dots, m$ , where  $P$  is the price of the contract. The cumulative investment funds all have the same

drift, which is  $r$ , and different volatilities, which are  $\sigma_{j1}^1$  and the total market value at time  $t$  is  $D_t^1 = \sum_{j=0}^n D_{jt}^1$ , where  $D_{jt}^1$ 's satisfy the stochastic differential equations

$$dD_{jt}^1 = rD_{jt}^1 dt + \sigma_{j1}^1 D_{jt}^1 dW_j^1 \tag{1}$$

for  $j = 0, 1, 2, \dots, m$ , where  $D_{jt}^1 = D_{j(t-\delta)}^1 - C_{tt} p_x (1+p)$ ,\*\*\* when  $t = m+1, \dots, n$ , and  $\delta$  is the infinite small increment,  $C_{tt} p_x (1+p)$  is the outflow of annual payment at the deterministic times  $t = m+1, \dots, n$  and  $r$ , the instantaneous investment return rate, follows the stochastic differential equation:

$$dr = \beta(\mu - r)dt + \sigma\sqrt{r}dW, \tag{2}$$

with expense being  $p$  percent of the annuity payment and with  $W$  being Wiener processes  $W_0^1, W_1^1, W_2^1, \dots, W_m^1$  whose instantaneous correlation coefficients are denoted by  $\rho_{i,k}^1$ , with  $i, k = 0, 1, 2, \dots, m+1$ . Furthermore,  $\sigma$  is the standard deviation of interest rates,  $\mu$  is the long-run equilibrium rate of interest, with the gap between the current and long-run equilibrium level of rates represented by  $\mu - r$ , and  $\beta$  is a measure of the sense of urgency exhibited in financial markets to close the gap, and gives the speed at which the gap is reduced, where the speed is expressed in annual terms (as the interest rates are annual).

We assume that there is no surrender option.

Additional notation used in this paper is listed below:

$\sigma_{j1}^1$  is the standard deviation of cumulative investment in mutual funds at time  $j$ ;

$EV_n$  is the sum of expected net present values of the insurer and customer;

$EVIC_n$  is the expected net present value of the insurer;

$EVC_n$  is the expected net present value of the customer;

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\*\*\*The authors thank the anonymous referees for helping clarify this point.

$PINV_n$  is the expected present value of net income of cumulative investment;

$p$  is the expense percentage of annuity payment;

$PI$  is the expected present value of premiums;

$PO$  is the expected present value of annuity payments;

$\gamma$  is the rate of the policyholder's participation in the investment returns;

$Aa$  is the level annuity benefit payable to the insured;

$AV_t$  is the variable annuity benefit payable to the insured;

$r_g$  is the minimum guaranteed return rate, and

$r_f$  is the risk-free force of interest.

## PRICING MODEL WITH LEVEL PREMIUM PAYMENT

With a participating contract, the benefit payable to the insured alive at time  $t$ , where  $t > m$ , is  $C_t = 1 + div_t$ , where  $div_t$  is the dividend received by the customer at time  $t$ . Here, we assume that the initial annuity payment is a monetary unit. In general, participating contracts have a minimum guaranteed return rate. We first assume that guaranteed rate to be equal to zero in order to simplify the analysis. In a later section, we will discuss the cases of minimum guaranteed return rate not equal to zero. Let  $\gamma$ , where  $0 < \gamma \leq 1$ , be the participation rate of the policyholder in the return generated by investment. From the perspective of this paper,  $\gamma$  is a form of price of this financial product. To simplify the analysis, we assume that the dividend originates only from the investment return and is distributed to the policyholder by increasing the annuity benefit. With the minimum rate of return guarantee, we obtain

$$div_t = \gamma \max(0, \Delta D_t^1) \quad (3)$$

where  $\Delta D_t^1 = D_t^1 - D_{t-1}^1$  is the investment return at the end of year  $t$ . Hence  $div_t$  is the payoff of a call option with exercise price zero. We use the force of interest to specify the discount factor  $v(t) = e^{-r_f t}$ , where  $r_f$  is the risk-free interest rate. The expected present value of benefit payable to the insured alive at time  $t$  is

$$AV_t = E^Q(C_t)v(t)_t p_x, \tag{4}$$

where  $t = m + 1, \dots, n$  and  $E^Q$  denotes the expectation operator under the equivalent martingale measure. Based on financial economics, there exists no arbitrage opportunity because the expected value of the change in value of the asset for the whole period is equal to zero. Therefore the measure described is a martingale measure.  $AV_t$  is also called the variable annuity benefit at time  $t$  payable to the insured alive. The fixed annuity benefit payable to an insured who is alive is

$$Aa = \frac{\sum_{t=m+1}^n E^Q(C_t)v(t)_t p_x}{\sum_{t=m+1}^n v(t)_t p_x}. \tag{5}$$

We assume that the insurer is risk neutral—that is, the insurer’s decision-making is based on the criterion of expected value. We here establish two objective functions, which maximize the insurer’s benefits and customer’s benefits respectively. Let  $PI$  be the present value of premiums,  $PO$  be the present value of annuity benefits paid to customers, and  $PINV$  be the expected present value of net income of cumulative investment (it also is the cumulative value of call options with different exercise terms and different exercise prices), then

$$PI = \sum_{t=0}^m P \cdot v(t)_t p_x, \tag{6}$$

$$PO = \sum_{t=m+1}^n E^Q(C_t)v(t)_t p_x, \tag{7}$$

$$PINV = \sum_{t=1}^n E^Q(\max(0, \Delta D_t^1))v(t) \tag{8}^{****}$$

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\*\*\*\*The authors thank the anonymous referees for helping clarify this point (see equation (8)).

$\Delta D_t^1$  is the incremental value of the financial portfolio of the insurer at time  $t$ ,  $C_t = 1 + div_t$ ,  $div_t = \gamma \max(0, \Delta D_t^1)$ .

Substituting  $v(t) = e^{-rt}$  into (4) and (5), we know that fixed annuity benefit is satisfied with

$$Aa = \frac{\sum_{t=m+1}^n e^{-rt} E^Q(C_t) \cdot {}_t p_x}{\sum_{t=m+1}^n A(t) e^{-rt} {}_t p_x}. \quad (10)$$

The actuarial value of the variable annuity at time  $t$  is

$$AV_t = e^{-rt} E^Q(C_t) \cdot {}_t p_x \quad (11)$$

for  $t = m+1, \dots, n$ . By substituting  $v(t) = e^{-rt}$  into equations (6), (7), (8), we get

$$PI = \sum_{t=0}^m P e^{-rt} {}_t p_x, \quad (12)$$

$$PO = \sum_{t=m+1}^n e^{-rt} E^Q(C_t) \cdot {}_t p_x, \quad (13)$$

where  $C_t = 1 + div$ ,  $div = \gamma \max(0, \Delta D_t^1)$ , and

$$PINV = \sum_{t=1}^n E^Q(\max(0, \Delta D_t^1)) e^{-rt}. \quad (14)$$

The insurer's benefits are maximized via the following optimization problem:

$$\text{Max} EVIC_n(\gamma) = PI - PO(1+p) + PINV, \quad (15)$$

subject to  $P > 0$ .

The optimization problem of the customer is

$$\text{Max}EVC_n(\gamma) = PO - PI, \quad (16)$$

subject to  $P > 0$ .

Generally, these two objectives cannot be realized at the same time, because the benefits of these two parties are in conflict. But if the insurer only considers profit maximization and captures the entire consumers' surplus, the customer will not have an incentive to establish the business relationship, and the insurer will not enjoy the profit after all. Therefore, the welfare available should be divided in a certain way, and a contract design should contain incentives aimed at resolving conflicts of interest between these two parties.

A realistic model of a process aimed at such a design is the bargaining model first proposed by Nash (1950), and extended by Thomson (1981). Although the customer cannot negotiate the price with the insurer, the customer can select the insurer. Thus, the insurer must forecast the customers' willingness to pay and create a pricing strategy that will be most likely to be accepted by the customer. The Nash (1950) solution maximizes the value of the product of the two utilities. The Thomson (1981) solution maximizes the sum of the two utilities. We replace the maximization of the sum of the two utilities with the maximization of the sum of the values to the two parties, subject to the constraints that the values of both parties not be less than zero, to determine a right division of the gain. Specifically, the optimization problem is:

$$\text{Max}EV_n(\gamma) = EVIC_n + EVC_n = PINV - PO \cdot p, \quad (17a)$$

subject to:

- (1)  $P > 0$ , (2)  $EVIC_n \geq 0$ , and (3)  $EVC_n \geq 0$ .

From equation (17a), we find that the optimal price is dependent on the main elements of competition, including the participating rate, expenses, and guaranteed rate of return. When the value of social welfare  $EV_n(\gamma)$  is greater than zero, the objective function has an optimal non-zero solution.

Substituting equation (13) and (14) into (17a), the equation (17a) becomes

$$\begin{aligned} \text{Max}EV_n(\gamma) &= EVIC_n + EVC_n = PINV - PO \cdot p & (17b) \\ &= \sum_{t=1}^n E^Q(\max(0, \Delta D_t^1))e^{-r_f t} - \sum_{t=m+1}^n E^Q(C_t)p \cdot e^{-r_f t} \end{aligned}$$

subject to:

(1)  $P > 0$ , (2)  $EVIC_n \geq 0$ , and (3)  $EVC_n \geq 0$ .

The advantage of the above model, as opposed to the classical cooperative game theory approach, is that it automatically takes the time value of money and the price of risk into account. In other words, it makes the power of insurance pricing in dealing with time and uncertainty available within the game theory model. In addition, since insurance companies and the insured share the common benefit, it will decrease insurance fraud. The decision parameter is the participation rate  $\gamma$ . Using Monte Carlo simulation and optimization techniques we can find the optimal solutions for the rate  $\gamma$ , price (i.e., level annual premium)  $P$ , fixed annuity  $Aa$ , and variable annuity  $AV_t$ , where  $t = m + 1, \dots, n$ . The optimization process is described below.

The complexity of the optimization problem, when considering multi-stage stochastic processes, does not allow for an explicit solution, but we can use numerical algorithms to seek a solution (for an approximated differentiation method please see the later section "Monte Carlo Simulation and Numerical Example"). The optimization problem we discuss here is a bounded constraint problem with boundary limits of  $0 < \gamma \leq 1$ . A search procedure is applied to locate the optimal solution  $\gamma$  which satisfies constraints (1)  $P > 0$ , (2)  $EVIC_n \geq 0$ , and (3)  $EVC_n \geq 0$ . For any random number  $\gamma$  with uniform distribution in the interval of  $[0,1]$ , we calculate the value of the objective function, compare the result with that calculated using the last random number, retain the larger result, and get rid of the smaller ones. After several thousand iterations, an approximate solution can be found.

## CALCULATION OF DEFAULT PROBABILITY AND VALUE AT RISK

Let  $VIC_n$  express the cumulative value of the insurance company at time  $n$ , and

$$\begin{aligned}
 VIC_n = & \sum_{t=1}^n \max(0, \Delta D_t^1) p_x e^{-r_f t} - \sum_{t=m+1}^n C_{tt} p_x (1+p) e^{-r_f t} \\
 & + \sum_{t=0}^m P_t p_x e^{-r_f t}.
 \end{aligned}
 \tag{18}$$

It is clear that  $VIC_n$  is a stochastic variable. It can be shown easily that the default probability is

$$Q(\gamma) = \Pr(VIC_n < 0). \tag{19}$$

With the help of simulation, we can find the empirical distribution of  $VIC_n/ EVIC_n$ , as well as the Value at Risk (VaR) at the confidence level  $\alpha$ . That is,

$$\text{VaR}(\gamma) = \frac{EVIC_n - VIC_{n,\alpha}}{EVIC_n}, \tag{20}$$

where  $EVIC_n$  is the expected company value with time length of  $n$  and  $VIC_{n,\alpha}$  indicates the lowest company value in the period of  $Y$  with confidence level  $\alpha$ , whose approximate value can be found by the average value of simulated sample values.  $EVIC_n$  and  $VIC_{n,\alpha}$  satisfy the following equation:

$$\Pr\left(\frac{VIC_n}{EVIC_n} \geq \frac{VIC_{n,\alpha}}{EVIC_n}\right) = 1 - \alpha. \tag{21}$$

## MONTE CARLO SIMULATION AND NUMERICAL EXAMPLE

In this section, we will use Monte Carlo simulation to get solutions for prices, insolvency probabilities, value at risk, and deferred annuities. In order to find solutions, we need to simulate the following stochastic processes:

$$dD_{jt}^1 = rD_{jt}^1 dt + \sigma_{j1}^1 D_{jt}^1 dW_j^1$$

for  $j = 0, 1, 2, \dots, m$ , where  $D_{jt}^1 = D_{j(t-\delta)}^1 - C_{tt}p_x(1+p)$ , when  $t = m+1, \dots, n$ ,  $\delta$  is the infinite small increment, and  $r$  satisfies the stochastic differential equation

$$dr = \beta(\mu - r)dt + \sigma\sqrt{r}dW,$$

with  $W_0^1, W_1^1, W_2^1, \dots, W_m^1, W$  being  $m+2$  Wiener processes and  $\rho_{i,k}^1$  for  $i, k = 0, 1, 2, \dots, m+1$ , expressing their instantaneous correlation coefficients. We let  $m = 4$ ,  $D_{j0}^1 = {}_j p_x P(0, j) P$   $j = 0, 1, 2, \dots, m$ ,

$$\rho^1 = \begin{bmatrix} 1 & 0.2 & 0.5 & 0.4 & 0.6 & 0.3 \\ 0.2 & 1 & 0.3 & 0.2 & 0.4 & 0.5 \\ 0.5 & 0.3 & 1 & 0.1 & 0.3 & 0.2 \\ 0.4 & 0.2 & 0.1 & 1 & 0.5 & 0.1 \\ 0.6 & 0.4 & 0.3 & 0.5 & 1 & 0.4 \\ 0.3 & 0.5 & 0.2 & 0.1 & 0.4 & 1 \end{bmatrix}$$

and  $n = 20$ . The initial value of investment return rate is set as  $r_0 = 0.03$ ,  $\sigma_1^1 = 0.03$ ,  $\mu = 0.05$ ,  $\sigma = 0.03$ ,  $x = 50$ ,  $\beta = 0.1$ ,  $n = 20$ ,  $p = 0.2$ ,  $r_f = 0.05$ , and the mortality data are from the Chinese Life Tables (1993). We use a basic methodology to simulate the underlying asset. We fix a time step  $\Delta t = 1$  and approximate the stochastic differential equations at time  $t$  by:

$$D_{j0}^1 = {}_j p_x P(0, j) P = {}_j p_x e^{-r_j t} P,$$

$$D_{jt}^1 = \begin{cases} D_{jt-1}^1(1+r+\sigma_1 \varepsilon_{jt} \rho_{jt} \sqrt{r}) & \text{when } t = 1, 2, \dots, 4 \\ [D_{jt-1}^1 - C_{t-1t-1} p_x(1+p)](1+r+\sigma_1 \varepsilon_{jt} \sqrt{r}) & \text{when } t = 5, \dots, 20, \end{cases} \quad (22)$$

for  $j = 0, 1, 2, 3, 4$ , where

$$C_{t-1} = 1 + div_{t-1} = 1 + \gamma \max(0, \Delta D_{t-1}^1),$$

$$\Delta D_{t-1}^1 = D_{t-1}^1 - D_{t-2}^1, \quad D_{t-1}^1 = \sum_{j=0}^n D_{jt-1}^1,$$

$$\Delta r = r_t - r_{t-1} = \beta(\mu - r_{t-1}) + \sigma_r \sqrt{r_{t-1}} \varepsilon, \quad (23)$$

$\varepsilon_{jt} \ j=0, 1, 2, 3, 4,$  and  $\varepsilon$  are random variables that follow a standard normal distribution. Given the functions of price, we can find optimal solutions. Assume the willingness to pay by customer is related to his attitude to risk and the function of price is

$$P = AA(P)(1 - e^{-k\gamma}), \quad (24)$$

where  $k$  is a risk-aversion constant,  $\gamma$  is the participation rate paid by the insurance company, and  $AA(P)$  is a coefficient related to price through the annuity benefit obtained by the customer  $Aa$ . Based on the customers' levels of risk aversion, we divide the customer group into three segments. We set  $k = 1, 10, 100$ , where  $k = 1$  corresponds to risk-loving customers,  $k = 10$  corresponds to customers with intermediate risk attitudes, and  $k = 100$  corresponds to risk-averse customers, and formulate three different objective functions with  $k = 1, 10,$  and  $100$ , respectively. By combining equations (17), (22), and (24), letting  $AA(P) = a + e^{-bAa(P)}$   $a = 1, b = 0.01$ , through iterating of limited times(s), and solving these three objective functions, we obtain the optimal solutions shown in Table 1 (for the proof of the existence of a derivative, and the maximum value and continuity of objective functions, please see the Appendix).

From Table 1, supported by the continuity and differentiability of the function analyzed given in the Appendix, we find that the annuity per unit price decreases with the increase of the coefficient  $k$  of risk aversion. That is, for the same payment, the risk-averse customer tends to be satisfied with a smaller benefit. The value at risk decreases with the increase in the coefficient of risk aversion. For the market segment of  $k = 1$ , the optimal solution does not exist. This group of customers expects to obtain a higher dividend and is not concerned about bearing higher credit risk. The best strategy for the insurer with respect to this group is not to sell policies to these customers, since the value at risk and default probability are too high and the profit of the insurer is negative.

**Table 1.** Optimal Solutions When  $k$  Changes

|   | $k = 1$             | $k = 10$ | $k = 100$ |
|---|---------------------|----------|-----------|
| $\max EV_n(\gamma^*)$                     | –                   | 14.5090  | 15.0409   |
| $\gamma^*$                                | No optimal solution | 0.50     | 0.10      |
| $P(\gamma^*)$                             | No optimal solution | 1.9724   | 2.0997    |
| $Q(\gamma^*)$                             | No optimal solution | 0        | 0         |
| Var( $\gamma^*$ ) (0.05 confidence level) | No optimal solution | 0.3108   | 0.0234    |
| $Aa(\gamma^*)$                            | No optimal solution | 1.4356   | 1.1108    |
| $Aa(\gamma^*)/P(\gamma^*)$                | –                   | 0.7278   | 0.5290    |

**Table 2.** Values of Variable Annuity

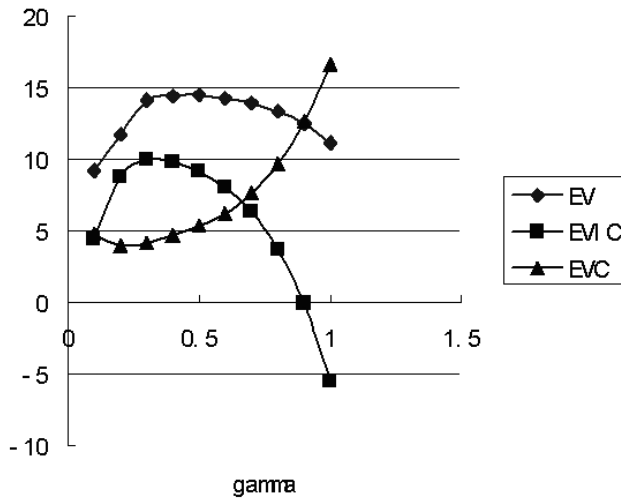
|        | $k = 10$  |        |        |        |        |        |        |        |
|--------|-----------|--------|--------|--------|--------|--------|--------|--------|
| $t$    | 6         | 7      | 8      | 9      | 10     | 11     | 12     | 13     |
| $AV_t$ | 1.2405    | 1.2710 | 1.3034 | 1.3396 | 1.3745 | 1.4135 | 1.4557 | 1.5012 |
| $t$    | 14        | 15     | 16     | 17     | 18     | 19     | 20     |        |
| $AV_t$ | 1.5418    | 1.6063 | 1.6583 | 1.7167 | 1.7854 | 1.8611 | 1.9442 |        |
|        | $k = 100$ |        |        |        |        |        |        |        |
| $t$    | 6         | 7      | 8      | 9      | 10     | 11     | 12     | 13     |
| $AV_t$ | 1.0903    | 1.1011 | 1.1136 | 1.1264 | 1.1390 | 1.1538 | 1.1707 | 1.1885 |
| $t$    | 14        | 15     | 16     | 17     | 18     | 19     | 20     |        |
| $AV_t$ | 1.2074    | 1.2260 | 1.2469 | 1.2713 | 1.2964 | 1.3234 | 1.3533 |        |

Table 2 displays the variable annuity benefits at time  $t$ , for  $t = 1, \dots, n$ , obtained by customers. Other assumptions for Table 2: For  $k = 10, \gamma = 0.50$ . For  $k = 100, \gamma = 0.1$ .

Table 3 and Figure 1 illustrate the relation between the dividend participation rate  $\gamma$  and expected insurance company value  $EVIC_n$ ,

**Table 3.** Change Pattern of  $EVIC$ ,  $EVC$ , and  $EV$  When  $k = 10$

|                  |         |         |         |         |         |         |         |         |
|------------------|---------|---------|---------|---------|---------|---------|---------|---------|
| $\gamma$         | 0.1     | 0.2     | 0.3     | 0.4     | 0.5*    | 0.6     | 0.7     | 0.8     |
| $EVIC_n(\gamma)$ | 4.4002  | 8.7644  | 9.9490  | 9.7727  | 9.1510  | 8.0382  | 6.3331  | 3.6570  |
| $EVC_n(\gamma)$  | 4.793   | 3.9606  | 4.1696  | 4.6786  | 5.358   | 6.207   | 7.593   | 9.6932  |
| $EV_n(\gamma)$   | 9.1932  | 11.7250 | 14.1186 | 14.4513 | 14.5090 | 14.2452 | 13.9261 | 13.3505 |
| $\gamma$         | 0.9     | 1.0     |         |         |         |         |         |         |
| $EVIC_n(\gamma)$ | -0.1148 | -5.5189 |         |         |         |         |         |         |
| $EVC_n(\gamma)$  | 12.6061 | 16.634  |         |         |         |         |         |         |
| $EV_n(\gamma)$   | 12.4913 | 11.1151 |         |         |         |         |         |         |



**Fig. 1.** The change pattern of  $EVIC$ ,  $EVC$ , and  $EV$  when  $k = 10$ .

expected value of consumer's benefits  $EVC_n$ , and the sum of the values of two parties  $EV_n$  when the coefficient of risk aversion is  $k = 10$ . Table 3 and Figure 1 indicate that the expected insurer's value reaches its maximum value at the point (0.3, 9.9490), the expected value of customer's benefits reaches its maximum value at the point (1.0, 16.634), and the sum of the benefits of both parties reaches its maximum value at the point (0.50, 14.5090), which appears to be a result of bargaining between the two

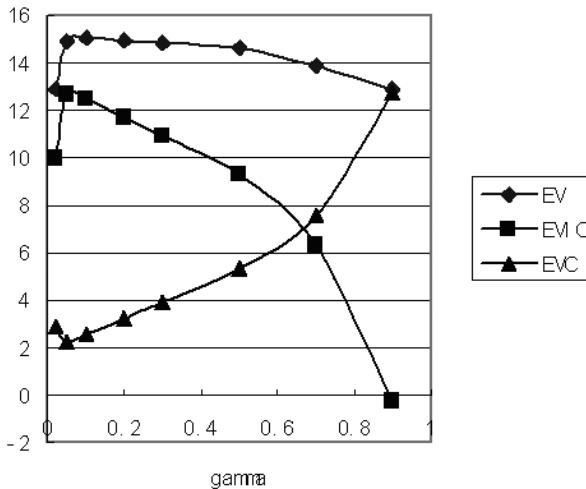


Fig. 2 The change pattern of  $EVIC$ ,  $EVC$ , and  $EV$  when  $k = 100$ .

parties. If the insurer simply starts off from the objective of being beneficial to itself, the insurer would like to choose the strategy  $\gamma = 0.3$ . However, this is not the best strategy of maximization of overall welfare, as the benefit of the customers is nearly at its lowest level and customers will likely not accept this strategy and not buy the policies. On the other hand, if the insurer simply starts off with the customers' preference, and chooses the strategy  $\gamma = 1$ , it is also not the best strategy of welfare maximization, because the insurer's profit reaches its lowest point and is even negative, and default probability reaches its highest point, with  $Q(\gamma = 1) = 1$ , and therefore the values of customers' benefits will definitely not be assured and they will also not accept this strategy. Therefore, the best strategy that can be accepted by both parties is to compromise by pursuing welfare maximization—i.e., maximizing the sum of the values of both parties, which is a very reasonable result: the insurer obtains a suitable return and the policyholder obtains the remaining return by paying premiums for the investment funds of the insurance company.

We reach a similar conclusion when we analyze the case of  $k = 100$  (see Figure 2 and Table 4). We find that the insurer's benefit reaches a maximum when  $\gamma = 0.05$  and the sum of both parties' benefits reaches a maximum when  $\gamma = 0.10$ . Therefore  $\gamma = 0.1$  is the best selection considering maximizing the sum of both parties' benefits under the constraint of the customers' value being positive. Compared with the case  $k = 10$ , the benefit obtained by the policyholder is much less because the policyholder is more risk averse.

**Table 4.** The change pattern of *EVIC*, *EVC*, and *EV* when  $k = 100$

| $\gamma$ | 0.02    | 0.05    | 0.1*    | 0.2     | 0.3     | 0.5    | 0.7     | 0.9     |
|----------|---------|---------|---------|---------|---------|--------|---------|---------|
| $EVIC_n$ | 9.9776  | 12.6338 | 12.4765 | 11.6758 | 10.9137 | 9.2787 | 6.3038  | -0.2708 |
| $EVC_n$  | 2.9065  | 2.2592  | 2.5644  | 3.2401  | 3.9188  | 5.3417 | 7.5506  | 12.7431 |
| $EV_n$   | 12.8841 | 14.8930 | 15.0909 | 14.9159 | 14.9325 | 14.62  | 13.8544 | 12.4723 |

\* The optimal solution.

**Table 5.** Change Patterns of Prices When  $\mu$  and  $p$  Change

|               | $\mu = 0.05$ |        |        |        |
|---------------|--------------|--------|--------|--------|
| $p$           | 0.15         | 0.20   | 0.25   | 0.30   |
| $\gamma^*$    | 0.56         | 0.50   | 0.40   | 0.32   |
| $P(\gamma)^*$ | 1.9926       | 1.9724 | 1.9635 | 1.9185 |
|               | $p = 0.20$   |        |        |        |
| $\mu$         | 0.05         | 0.06   | 0.07   | 0.08   |
| $\gamma^*$    | 0.50         | 0.54   | 0.57   | 0.60   |
| $P(\gamma)^*$ | 1.9724       | 1.9890 | 1.9926 | 1.9950 |

\* The optimal solution.

## ANALYSIS OF THE INFLUENCE OF ELEMENTS OF COMPETITION ON THE OPTIMAL PRICE

Since the competitive power of insurance companies affects the optimal price level, in this section, we will analyze the effect on the optimal price when the elements of competition including expenses and the rate of return change. Table 5 lists the results.

From Table 5 we find that the optimum price decreases with an increase in expenses. This result is natural since the willingness to pay by the customer decreases when the expenses of the insurance companies rise. We also find that the optimum price is positively related to the long-term interest rate; that is to say, the greater the long-term interest rate is, the greater the customers' willingness to pay is.

**Table 6.** VaR Values When  $\gamma$  and  $r_g$  Take Different Values

|               |         |         |         |         |         |         |         |
|---------------|---------|---------|---------|---------|---------|---------|---------|
| $\gamma$      | 0.1     | 0.2     | 0.3     | 0.4     | 0.56*   | 0.6     | 0.7     |
| $r_g = -0.01$ | -0.7316 | 9.2026  | 12.7880 | 14.0571 | 14.5262 | 14.2379 | 13.7685 |
| $\gamma$      | 0.1     | 0.2     | 0.3     | 0.4     | 0.5     | 0.6     | 0.7     |
| $r_g = 0$     | 9.1932  | 11.7250 | 14.1186 | 14.4513 | 14.5090 | 14.2452 | 13.9261 |
| $\gamma$      | 0.1     | 0.2     | 0.3     | 0.4*    | 0.5     | 0.6     | 0.7     |
| $r_g = 0.01$  | 12.6856 | 13.9934 | 14.4289 | 14.5732 | 14.4329 | 14.3495 | 14.0959 |
| $\gamma$      | 0.1     | 0.2     | 0.3*    | 0.4     | 0.5     | 0.6     | 0.7     |
| $r_g = 0.03$  | 14.3044 | 14.4467 | 14.5220 | 14.4620 | 14.4019 | 14.2675 | 14.2314 |

\* The optimal solution.

## PARTICIPATING ANNUITY WITH MINIMUM GUARANTEED RETURN RATE

Let the minimum guaranteed return rate be  $r_g$ , then the benefit payable to the insured alive at time  $t$ , where  $t > m$ , is

$$C_t = 1 + D_{t-1}^1 r_g + \gamma \max(0, \Delta D_t^1 - D_{t-1}^1 r_g).$$

Let  $P = AA(P)(1 - e^{-(k_1\gamma + k_2r_g)})$ . Table 6 lists the results of the sum of the benefits for both sides to the contract when  $\gamma$  and  $r_g$  take on different values. From Table 6 we find that the optimal solution for the participation rate  $\gamma$  decreases with an increase in the minimum guaranteed return rate  $r_g$ . This result is natural since, if the policyholder requires a guaranteed benefit, it means the policyholder should give up some dividend.

Figure 3 illustrates the relationship between the participation rate  $\gamma$  and the Value at Risk at the 0.05 confidence level, VaR, with the minimum guarantee return rate  $r_g$  at three levels (-0.01, 0, and 0.01). From Figure 3 we see that the values of VaR first decrease then increase. The levels of Value at Risk are lower when the participation rates take the values of the optimal solutions, which illustrates that using the criterion of maximizing

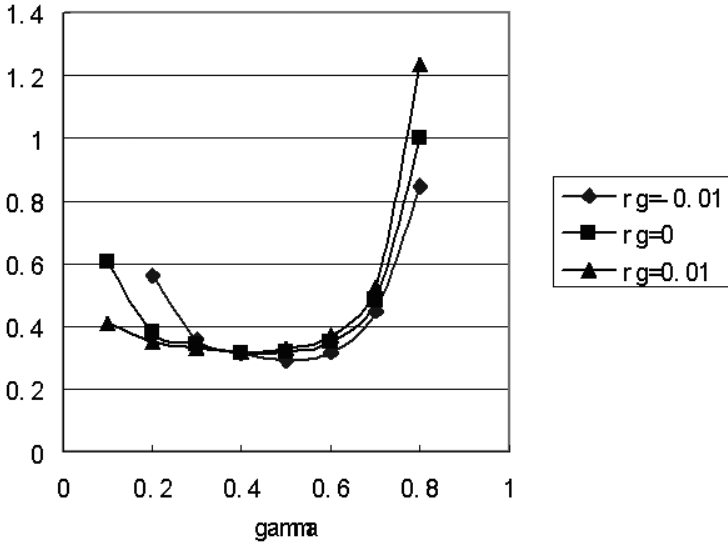


Fig. 3. VaR values when  $\gamma$  and  $r_g$  take different values

the social benefit to determine the optimal participation rate, both sides of the contract not only can obtain satisfying benefits, but also may bear smaller risks. For most cases (except in the cases when the participation rate takes smaller values), the Value at Risk increases with an increase in the minimum guaranteed rate of return, which means that if the policyholder requires more return guarantee, he must bear more credit risk.

### CONCLUSIONS

In this paper, we discuss the pricing of a participating deferred annuity using game theory. The established objective functions are based on maximizing welfare, which is the sum of the expected net present values of the insurer and the customer, which are functions themselves of the dividend ratio  $\gamma$ . Different price functions are assumed according to different risk attitudes of customers. The rate of return and multi-stage cumulative investments are thought of as correlated stochastic processes. Monte Carlo simulation and stochastic optimization technique are used to find optimal solutions and analyze the equilibrium conditions. Usually, a deferred annuity is a long-term contract, and if offering a surrender option, it should be more valuable to the customer. Further research will focus on studying the surrender value of participating deferred annuities.

### APPENDIX

The objective function (17b) is

$$EV_n(\gamma) = \sum_{t=1}^n E^Q(\max(0, \Delta D_t^1))e^{-r_f t} = \sum_{t=m+1}^n E^Q(C_t)p e^{-r_f t}$$

where  $C_t = 1 + div_t$  and  $div_t$  is the dividend received by the customer at time. By using equation (22), and iterating repeatedly, the equation

$$D_t^1 = \sum_{j=0}^n D_{jt}^1$$

can be expressed as an explicit function of price  $P(\gamma)$ .

Therefore, we can let

$$D_t^1 = H(P(\gamma))G(t), \quad \Delta D_t^1 = H(P(\gamma))\Delta G(t),$$

where  $G(t)$  is a function that is not related to  $\gamma$  and  $C_t$  can be expressed as  $C_t = 1 + \gamma \max(0, HP(\gamma))(\Delta G(t))$ .

$$\frac{d(\Delta D_t^1)}{d\gamma} = \frac{d(\Delta D_t^1)}{dP} \frac{dH}{dP} \frac{dP}{dAA} \frac{dAA}{dP} \frac{dP}{d\gamma} = \Delta G(t) \frac{dH}{dP} \frac{dP}{dAA} \frac{dAA}{dP} \frac{dP}{d\gamma},$$

$$\frac{dC_t}{d\gamma} = \begin{cases} 0, & \text{when } \Delta D_t^1 < 0, \\ H(P(\gamma))(\Delta G(t)) + \gamma \frac{d(\Delta D_t^1)}{dP} \frac{dH}{dP} \frac{dP}{dAA} \frac{dAA}{dP} \frac{dP}{d\gamma} & \\ = H(P(\gamma))\Delta G(t) + \gamma \Delta G(t) \frac{dH}{dP} \frac{dP}{dAA} \frac{dAA}{dP} \frac{dP}{d\gamma} & \text{otherwise.} \end{cases}$$

and

$$\frac{dEV_n(\gamma)}{d\gamma} = \begin{cases} 0 & \text{when } (\Delta D_t^1 < 0) \\ \left( \sum_{t=1}^n e^{-r_f t} E^Q(\Delta G(t)) \right) \left( \frac{dH}{dP} \frac{dP}{dAA} \frac{dAA}{dP} \frac{dP}{d\gamma} \right) \\ \left( \sum_{t=m+1}^n j p_x \cdot p e^{-r_f t} \left( H(P(\gamma)) E \Delta G(t) + \gamma \Delta G(t) \right) \frac{dH}{dP} \frac{dP}{dAA} \frac{dAA}{dP} \frac{dP}{d\gamma} \right) \end{cases}$$

otherwise.

Setting  $P(\gamma) = AA(P)(1 - e^{-k\gamma})$ , we see that the derivative  $\frac{dH}{dP}, \frac{dP}{dAA}$ ,  $\frac{dAA}{dP}, \frac{dP}{d\gamma}$  exists and  $\frac{dEV_n}{d\gamma}$  exists too; therefore, the function is continuous.

Similarly, we can prove that when the minimum guaranteed return rate is not equal to zero, the objective function is also continuous. Therefore, the maximum value exists in the interval of [0,1]. Similarly, we can prove that the functions  $EVIC_n(\gamma)$  and  $EVC_n(\gamma)$  are continuous and their maximum values are attained.

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